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THE ASYMMETRICAL MIXED TEMPERATURE PROBLEM FOR A TRANSVERSELY ISOTROPIC ELASTIC LAYER†

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The problem of the uniform heating of a two-layer plate is solved. The transversely isotropic elastic layer (soft plate) investigated is in ideal contact with an absolutely rigid layer, deformable only by thermal expansion. The generalized plane temperature problem reduces to determining the stress-strain state of the soft anisotropic layer investigated using the equations of the mixed problem of elasticity theory. At the ends of the boundary layer of the soft plate (a thin contact layer), no conditions are imposed. On the remaining part of the ends of the soft plate, the boundary conditions correspond to a free boundary. The problem has a bounded smooth solution. Unlike the approach described earlier [1], it is proposed to seek an accurate solution in the form of ordinary Fourier series with respect to a single longitudinal coordinate. Solutions in polynomials are also used. It is shown that the existence of these solutions in polynomials enables the convergence of the Fourier series to be improved considerably. © 2002 Elsevier Science Ltd. All rights reserved.

Earlier [1], a modification of Mathieu's method was proposed to solve the symmetrical temperature problem for a transversely isotropic elastic layer.

1. THE METHOD OF SOLUTION

The problem of the uniform heating of a two-layer plate is solved. The transversely isotropic elastic layer (soft plate) investigated is in ideal contact with an absolutely rigid layer, deformable only by thermal expansion. It is assumed that the layer of anisotropic material under investigation has practically no effect on the other layer by virtue of its relatively low stiffness.

The generalized plane temperature problem reduces to determining the stress–strain state of a transversely isotropic soft plate of length $2L$ and thickness H

$$|x'| \leq L, \quad 0 \leq y' \leq H$$

on the basis of the equations of the mixed problem of elasticity theory.

Generally speaking, the soft plate is bounded along the aix, υ, perpendicular to the axes directed along the length and thickness of the plate.

For the strain along this axis we have

$$\varepsilon_\upsilon = \lambda_0 T \tag{1.1}$$

where λ_0 is the coefficient of thermal expansion of the isotropic absolutely rigid layer, and T = const is the temperature increment.

Below, we will use dimensionless Cartesian coordinates x, y referred to L. Then $y = 0$ is the side surface of the layer investigated, $x = \pm 1$ are its ends, and $y = h$ is the contact surface with the absolutely rigid layer ($h = H/L$).

We will write the relation between the stresses σ_x, σ_y and σ_{xy} and the strains ε_x, ε_y and $\varepsilon_{xy}/2$ of the transversely isotropic material investigated for the case of generalized plane strain taking (1.1) into account

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$$E\varepsilon_x = \sigma_x - \nu_0\sigma_y + E(1+\nu)\Delta\lambda T + E\lambda_0 T \tag{1.2}$$

$$E\varepsilon_y = \omega\sigma_y - \nu_0\sigma_x + E[\lambda_y + \nu_0(1-\nu)\Delta\lambda]T, \quad E\varepsilon_{xy} = \gamma_0\sigma_{xy}$$

$$E = \frac{E_x}{1-\nu^2}, \quad \omega = \frac{k-(k\nu')^2}{1-\nu^2}, \quad \nu_0 = \frac{k\nu'}{1-\nu}, \quad k = \frac{E_x}{E_y}$$

$$\gamma_0 = \frac{\gamma}{1-\nu^2}, \quad \gamma = \frac{E_x}{G}, \quad \Delta\lambda = \lambda_x - \lambda_0$$

Here it is assumed that the axis of isotropy (symmetry) of the material is directed along the y axis, E_x and E_y are the elastic moduli along the x and y axes, G is the shear modulus for the (x, y) plane, ν and ν' are Poisson's ratios, and λ_x and λ_y are the coefficients of thermal expansion along the x and y axes respectively. Poisson's ratio ν characterizes the transverse compression in the plane of anisotropy (x, υ) for extension in this plane, and ν' is the same, but for extension in the direction of the y axis [2].

The stress σ_υ is determined from Eq. (1.1)

$$\sigma_\upsilon = \nu\sigma_x + k\nu'\sigma_y - E_x\Delta\lambda T$$

Taking relations (1.1) and (1.2) into account, we will write the equations of the anisotropic theory of elasticity in the form

$$\omega\frac{\partial^4 F}{\partial x^4} + \mu\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{1.3}$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad \mu = \gamma_0 - 2\nu_0$$

The boundary conditions on the side surface $y = 0$ and on the contact surface $y = h$ have the form

$$y = 0: \ \sigma_y = \sigma_{xy} = 0 \tag{1.4}$$

$$y = h: \ \varepsilon_x = \lambda_0 T, \quad \partial W/\partial x = 0 \tag{1.5}$$

where W is the dimensionless displacement (referred to L) along the y axis.

The last relation in Eqs (1.5) reduces to the equation

$$y = h: \ \int_0^x \frac{\partial \sigma_x}{\partial y} dx = (\mu + \nu_0)\sigma_{xy}$$

On the ends of the layer investigated there are no loads

$$x = \pm 1: \sigma_x = \sigma_{xy} = 0 \tag{1.6}$$

The solution of problem (1.1)–(1.6) presumably has a singularity (infinite) at the corner points $(x = \pm 1, y = h)$ of the anisotropic layer investigated. We will find the bounded smooth stress–strain state of this layer, which is identical everywhere with the solution of system (1.1)–(1.6), with the exception of a certain small region at the corner points $(x = \pm 1, y = h)$.

The problem can be formulated as follows.

We will conventionally divide the layer investigated into strips S_n: $\{y_n \geqslant y \geqslant y_{n-1}\}$, $n = 1, 2, \ldots, N$, where $y_0 = 0, y_N = h$, and N is the total number of strips.

The plate S_N is a thin contact layer of *a priori* specified small thickness $\sigma_N \ll h$. In this layer it is required to find an accurate solution of Eq. (1.3) that corresponds to the internal mixed temperature problem, i.e. this solution should satisfy only the boundary conditions on the side surfaces $y = h - \sigma_N$ and $y = h$. This layer will be called the boundary layer [1].

In the other layers it is required to find an accurate solution of the temperature problem that makes it possible to satisfy the corresponding boundary conditions on the side surfaces $y = y_{n-1}$, $y = y_n$ $(n = 1, 2, \ldots, N-1)$ and the integral boundary conditions on the ends $x = \pm 1$, which correspond to the free boundary. Generally speaking, these layers can have different thicknesses $\delta_n = y_n - y_{n-1}$.

The exact solution of the problem has the form

$$\begin{aligned}
\sigma_x^{(n)} &= M_6^{(n)}(6x^2\xi_n^2-\mu\xi_n^4)+N_6^{(n)}(x^4-1-\omega\xi_n^4)+M_4^{(n)}(x^2-1-\mu\xi_n^2)-\\
&-N_4^{(n)}\omega\xi_n^2+M_2^{(n)}+D_5^{(n)}(3x^2\xi_n-\mu\xi_n^3)+D_3^{(n)}\xi_n+\Sigma_1\\
\sigma_y^{(n)} &= M_6^{(n)}\left[\xi_n^4-\left(x^4-\frac{1}{5}\right)\frac{1}{\omega}\right]+N_6^{(n)}\left[6x^2\xi_n^2-\frac{\mu}{\omega}\left(x^4-\frac{1}{5}\right)\right]+\\
&+M_4^{(n)}\xi_n^2+N_4^{(n)}\left(x^2-\frac{1}{3}\right)-D_1^{(n)}\xi_n+D_5^{(n)}\xi_n^3-\Sigma_2 \qquad (1.7)\\
\sigma_{xy}^{(n)} &= -4M_6^{(n)}x\xi_n^3-4N_6^{(n)}x^3\xi_n-2M_4^{(n)}x\xi_n+D_1^{(n)}x-3D_5^{(n)}x\xi_n^2+\\
&+\sum_{m=1}^{\infty}\sin(\pi mx)\cdot\sum_{i=1}^{2}[R_{2i-1,m}^{(n)}\operatorname{ch}(\alpha_{i,m}\xi_n)+R_{2i,m}^{(n)}\operatorname{sh}(\alpha_{i,m}\xi_n)]\varkappa_i
\end{aligned}$$

where

$$\Sigma_l=\sum_{m=1}^{\infty}\cos(\pi mx)\cdot\sum_{i=1}^{2}[R_{2i-1,m}^{(n)}\operatorname{sh}(\alpha_{i,m}\xi_n)+R_{2i,m}^{(n)}\operatorname{ch}(\alpha_{i,m}\xi_n)][l-1+(2-l)\varkappa_i^2]$$

$$\alpha_{i,m}=\pi m\varkappa_i,\quad \xi_n=y-y_{n-1},\quad y_0=0,\quad y_N=h,\quad 0\leqslant\xi_n\leqslant\delta_n$$

$$2\varkappa_i^2=\mu-(-1)^i(\mu^2-4\omega)^{1/2},\quad i=1,\ 2,\quad \varkappa_i>0$$

Here, $\sigma_\alpha^{(n)}=\sigma_\alpha^{(n)}(x,y)$, while $R_{jm}^{(n)}(j=1,2,3,4)$, $M_{2j}^{(n)}(j=1,2,3)$ and $N_{2j}^{(n)}(j=2,3)$, and $D_1^{(n)}$, $D_3^{(n)}$ and $D_5^{(n)}$ are constants to be determined during the solution.

For a transversely isotropic material with pronounced anisotropy we can assume that $\mu>2\sqrt{\omega}$.

For the boundary layer we assume that

$$M_6^{(N)}=N_6^{(N)}=0 \qquad (1.8)$$

The conditions of conjugation of the layers will be written in the form

$$y=y_{n-1}:\quad \sigma_x^{(n)}=\sigma_x^{(n-1)},\quad \sigma_y^{(n)}=\sigma_y^{(n-1)} \qquad (1.9)$$

$$y=y_{n-1}:\quad \sigma_{xy}^{(n)}=\sigma_{xy}^{(n-1)},\quad \int_0^x\frac{\partial\sigma_x^{(n)}}{\partial y}dx=\int_0^x\frac{\partial\sigma_x^{(n-1)}}{\partial y}dx \qquad (1.10)$$

$$n=2,3,\ldots,N$$

The first equation in (1.9) follows from the continuity of the strain ε_x along the y axis. The second equation in (1.10) indicates that the quantity $\partial W/\partial x$ is continuous along the y axis.

Boundary condition (1.5) on the contact surface of the soft anisotropic layer with the rigid plate takes the form

$$y=h:\quad \varepsilon_x^{(N)}=\lambda_0T,\quad \int_0^x\frac{\partial\sigma_x^{(N)}}{\partial y}dx=(\mu+\nu_0)\sigma_{xy}^{(N)} \qquad (1.11)$$

We will rewrite condition (1.4) on the free surface

$$y=0:\quad \sigma_y^{(1)}=\sigma_{xy}^{(1)}=0 \qquad (1.12)$$

We will write the integral boundary conditions on the ends $x=\pm1$ taking into account the symmetry of the stresses with respect to the x coordinate

$$\int\sigma_x^{(n)}(1,\ y)dy=\int\sigma_x^{(n)}(1,\ y)\xi_ndy=\int\sigma_{xy}^{(n)}(1,\ y)dy=0 \qquad (1.13)$$

$$n=1,2,\ldots,N-1$$

In (1.13), the upper and lower limits of integration are respectively y_n and y_{n-1}.

From relations (1.7) it follows that the third condition of (1.13) is automatically satisfied.

The constants written above in formulae (1.7) are determined from Eqs (1.8)–(1.13).

The existence of particular solutions in polynomials of Eq. (1.3) enables us to improve the convergence of the Fourier series in (1.7).

We will briefly indicate the method used to set up an infinite system of algebraic equations for determining the required constants.

The following expansions of the functions in Fourier series are used

$$2x^2 - x^4 = \frac{7}{15} + 48 \sum_{m=1}^{\infty} (-1)^m \frac{\cos(\pi m x)}{(\pi m)^4}$$

$$x^3 - x = 12 \sum_{m=1}^{\infty} (-1)^m \frac{\sin(\pi m x)}{(\pi m)^3} \tag{1.14}$$

Functional equations (1.9), the first equation of (1.11) and the first equation of (1.12) are expanded in terms of the basis functions x^2, 1 and $\cos(\pi m x)$. This means that, in these equations, the polynomial $2x^2 - x^4$ according to the first formula of (1.14) is expanded in a Fourier series. Then, the algebraic expressions with factors $\cos(\pi m x)$ and x^2 and also the sum of all the constants (factor unity) of the given functional equation are equated to zero.

The functional equations (1.10), the second equation of (1.11) and the second equation of (1.12) are expanded in terms of the basis functions x and $\sin(\pi m x)$. Here, expansion of the function $x^3 - x$ in a Fourier series is used.

The system of algebraic equations obtained for determining the required constants in solution (1.7) is completed by Eqs (1.8) and the first two equations of (1.13).

2. RESULTS OF CALCULATIONS

Calculations of the dimensionless stresses (referred to the quantity $E\Delta\lambda T$), given in this section, are carried out with

$$k = 3, \gamma = 6, \nu = 0{,}2, \nu' = 0{,}1, h = 0{,}2, N = 6, \delta_N = 0{,}2h$$

$$\delta_i = \delta(i = 2,\ 3,\ 4,\ 5),\quad \delta_1 = 0{,}5\delta,\quad \delta = (h - 0{,}2h)/(N - 1{,}5),\quad L = 80$$

where L is the number at which the Fourier series is terminated along the x coordinate. The thickness of the first layer δ_1 is taken to be less than the thicknesses of the other layers in order to improve the approximation of the boundary condition $\sigma_x(1, y) = 0$ close to the free side surface $y = 0$.

The figure shows the distribution of the dimensionless stresses p_x (the continuous curve), p_y (the dot-dash curve) and p_{xy} (the dashed curve) along the x axis in different sections of the anisotropic layer investigated. Curves 1, 2, 3 and 4 correspond to the sections

$$y = 0,\quad y = 4h/9,\quad y = h_1,\quad y = h(h_1 = h - \delta_N = 0{,}8h)$$

Comparison of the results of a calculation of the stresses with indices (n) and $(n-1)$ at the interfaces of these solutions $y = y_{n-1}$ $(n = 2, 3, 4, 5, 6)$ indicates a high degree of convergence of the Fourier series in solution (1.7). In particular, the values of the stresses $p_y^{(n)}$ and $p_y^{(n-1)}$ at the interfaces $y = y_{n-1}$, $n = 4$, $6(y_3 = 4h/9, y_5 = h_1)$ are given below:

x	0	0.5	0.8	0.9	1
$10^4 \times p_y^{(6)}$	165	429	467	−700	−7760
$10^4 \times p_y^{(5)}$	166	430	468	−700	−7780
$10^4 \times p_y^{(4)}$	55	154	34	−697	460
$10^4 \times p_y^{(3)}$	55	154	34	−697	456

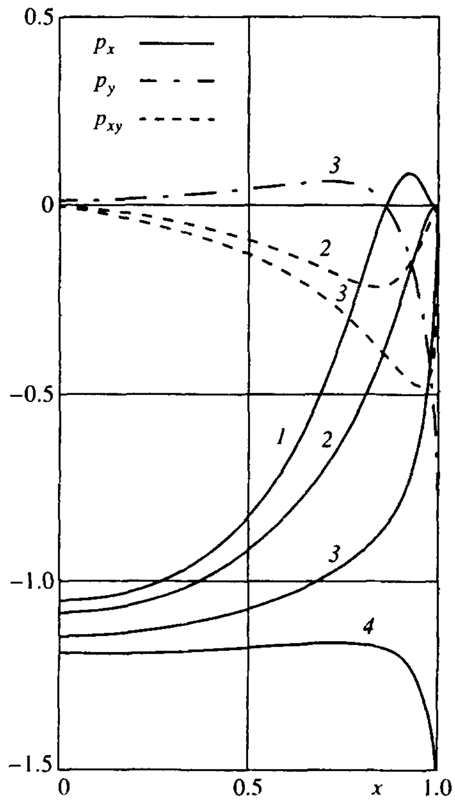

Fig. 1

The calculations also show that, when $N \geqslant 4$, the solutions are practically identical.

Earlier [1], the generalized plane problem of the uniform heating of a symmetrical three-layer plate with absolutely rigid outer layers deformable only by thermal expansion was solved. Similarly, a boundary layer of the investigated soft filler of transversely isotropic material was introduced. To find a symmetrical accurate solution of the problem, a modification of Mathieu's method was used; the solution was constructed as the superposition of ordinary Fourier series along the two coordinates x, y and partial solutions in polynomials of Eq. (1.3). This problem was also solved by the method described in this paper ($N = 5$, $\delta_i = \delta$, $i = 1, 2, 3, 4$). Comparison of the solutions with

$$k = 3, \ \gamma = 6, \ \nu = 0{,}2, \ \nu' = 0{,}1, \ h = 0{,}2, \ \delta_N = h/6$$

where $2h$ is the thickness of the anisotropic filler, and δ_N is the thickness of the boundary layer, showed that they are practically identical everywhere, including at the corner points $(\pm 1, h)$.

Note that the nature of the change along x in the magnitudes of the stresses obtained for the symmetrical and the asymmetrical temperature problems is approximately the same. The exception is the change in the stresses σ_x in a small neighbourhood of the point $x = 1, y = 0$. For example, continuous curve 1 in Fig. 1 has a zone of small tensile stresses close to the end $x = 1$.

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